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Kelvin–Helmholtz wave growth on cylindrical sheets

By G. D. CRAPPER,† N. DOMBROWSKI
AND G. A. D. PYOTT
University of Leeds

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A first-order analysis of Kelvin–Helmholtz wave growth on cylindrical sheets is carried out. It is demonstrated that the growth rate of both symmetric and antisymmetric waves increases significantly with reduction of the radius of the core.

A number of analytical studies have been made of Kelvin–Helmholtz wave growth on thin liquid sheets (Squire 1953; Hagerty & Shea 1955; Clark & Dombrowski 1972), but thus far these have been confined to flat interfaces. In an important type of atomizer, the swirl spray nozzle, the liquid emerges in the form of a conical attenuating sheet. This system is at present mathematically intractable, but a study of the simpler case of a cylindrical sheet should provide an indication of the effect of lateral curvature on wave growth. To this end, a linear analysis is carried out and the result compared with that for the flat system treated by Squire.

Recent studies (Crapper *et al.* 1973; Crapper, Dombrowski & Jepson 1975) have shown the viscosity of the gas phase to have a significant effect on wave growth. However, inviscid analyses provide a good description of the overall wave characteristics, and this approach is adopted here. We consider a cylindrical sheet with internal radius a and thickness h. In the undisturbed state the liquid in the sheet, of density ρ_L, has velocity V and the surrounding gas, of density ρ_G, is at rest. For incompressible irrotational flow the motion is defined by the Laplace equation

$$\nabla^2\phi_i = 0, \tag{1}$$

where $i = 1$, 0 and 2 correspond to the three regions indicated in figure 1.

Following the usual procedure, the displacements of the inner and outer surfaces may be taken respectively to be

$$\eta_1 = \epsilon\alpha \exp[i(kx-\omega t)], \quad \eta_2 = \epsilon\beta \exp[i(kx-\omega t)], \tag{2}$$

and to satisfy (1), the velocity potentials in cylindrical polar form (r, θ, x) for the three regions are assumed to be

$$\left.\begin{aligned} \phi_1 &= \epsilon R_1(r)\exp[i(kx-\omega t)], \\ \phi_0 &= \epsilon R_0(r)\exp[i(kx-\omega t)] + Vx, \\ \phi_2 &= \epsilon R_2(r)\exp[i(kx-\omega t)], \end{aligned}\right\} \tag{3}$$

† Present address: Department of Applied Mathematics and Theoretical Physics, University of Liverpool.

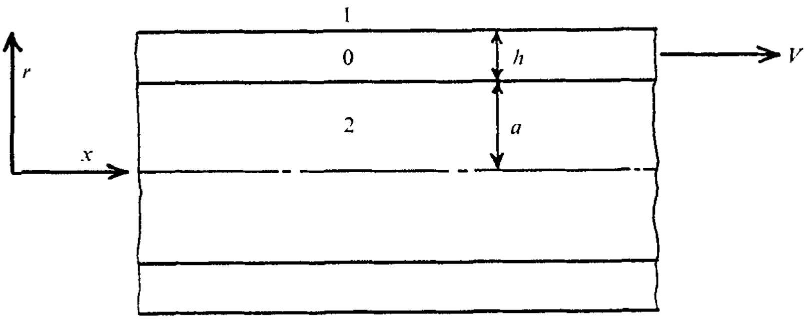


FIGURE 1. Flow system studied.

where

$$R_1 = A_1 K_0(kr), \quad R_0 = A_0 K_0(kr) + B_0 I_0(kr), \quad R_2 = A_2 I_0(kr)$$

and I_0 and K_0 are zero-order Bessel functions.

Retaining terms in ϵ only, (1) is solved by applying the Bernoulli equation at the boundaries,

$$\rho_L\left(\frac{\partial\phi_0}{\partial t} + V\frac{\partial\phi_0}{\partial x}\right) - \rho_G\frac{\partial\phi_1}{\partial t} - \sigma\left(\frac{\partial^2\eta_1}{\partial x^2} + \frac{\eta_1}{a^2}\right) = 0 \quad \text{at} \quad r = a+h, \tag{4}$$

$$\rho_G\frac{\partial\phi_2}{\partial t} - \rho_L\left(\frac{\partial\phi_0}{\partial t} + V\frac{\partial\phi_0}{\partial x}\right) - \sigma\left(\frac{\partial^2\eta_2}{\partial x^2} + \frac{\eta_2}{a^2}\right) = 0 \quad \text{at} \quad r = a, \tag{5}$$

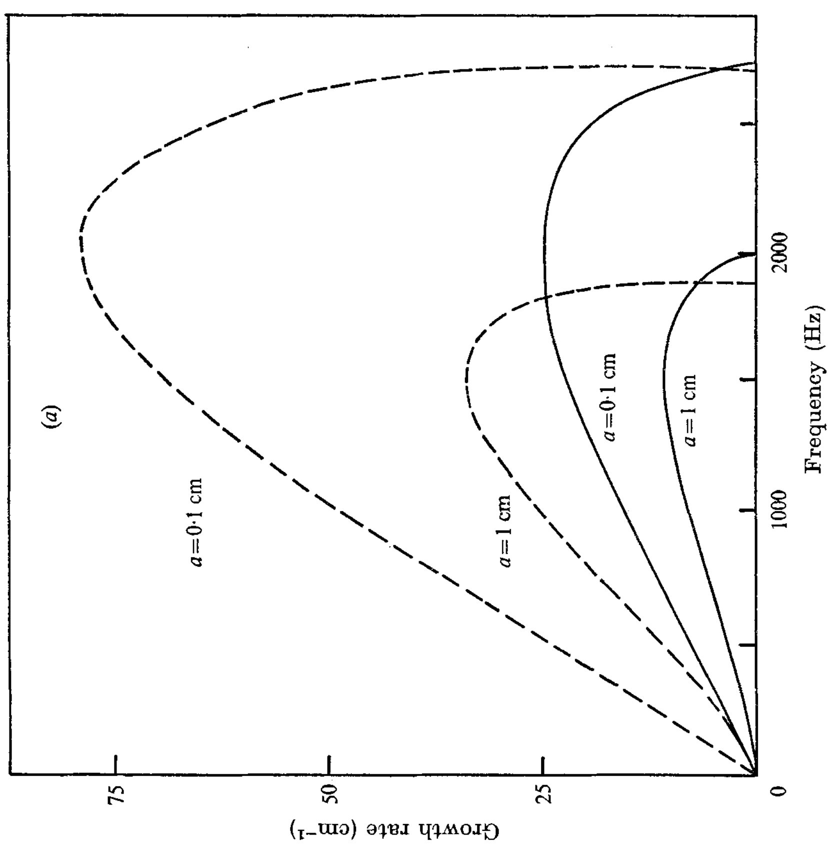
where σ = surface tension, and noting that the surface of the fluid moves with the fluid,

$$\frac{\partial\eta}{\partial t} + V\frac{\partial\eta}{\partial x} - \frac{\partial\phi_i}{\partial r} = 0 \quad \begin{cases}\text{at} & r = a \quad \text{for} \quad i = 0, 2,\\ \text{at} & r = a+h \quad \text{for} \quad i = 0, 1;\end{cases} \tag{6}$$

the V term only appears when $i = 0$.

Thus eliminating A_0, A_1, B_0 and B_1 and solving for α and β gives

$$\begin{aligned}
&\omega^4\left[(PQ-RS)\rho_L^2 + \frac{\rho_L\rho_G}{k}\left(\frac{K_0(k(a+h))}{K_1(k(a+h))}Q + \frac{I_0(ka)}{I_1(ka)}P\right) + \frac{\rho_G^2}{k^2}\frac{I_0(ka)}{I_1(ka)}\frac{K_0(k(a+h))}{K_1(k(a+h))}\right]\\
&-\omega^3\left[4kV\rho_L^2(PQ-RS) + 2\rho_L\rho_G V\left(\frac{K_0(k(a+h))}{K_1(k(a+h))}Q + \frac{I_0(ka)}{I_1(ka)}P\right)\right]\\
&+\omega^2\left[6\rho_L^2k^2V^2(PQ-RS) + \rho_L\rho_G kV^2\left(\frac{K_0(k(a+h))}{K_1(k(a+h))} + \frac{I_0(ka)}{I_1(ka)}P\right)\right.\\
&\left.+\rho_L\sigma\left(\frac{1}{a^2}-k^2\right)(P+Q) + \frac{\sigma\rho_G}{k}\left(\frac{K_0(k(a+h))}{K_1(k(a+h))} + \frac{I_0(ka)}{I_1(ka)}\right)\left(\frac{1}{a^2}-k^2\right)\right]\\
&-\omega\left[(4\rho_L^2k^3V^3(PQ-RS) + 2\rho_L kV\left(\frac{1}{a^2}-k^2\right)(P+Q)\right]\\
&+\rho_L^2k^4V^4(PQ-RS) + \sigma\rho_L k^2V^2\left(\frac{1}{a^2}-k^2\right)(P+Q) + \sigma^2\left(\frac{1}{a^2}-k^2\right)^2 = 0,
\end{aligned} \tag{7}$$



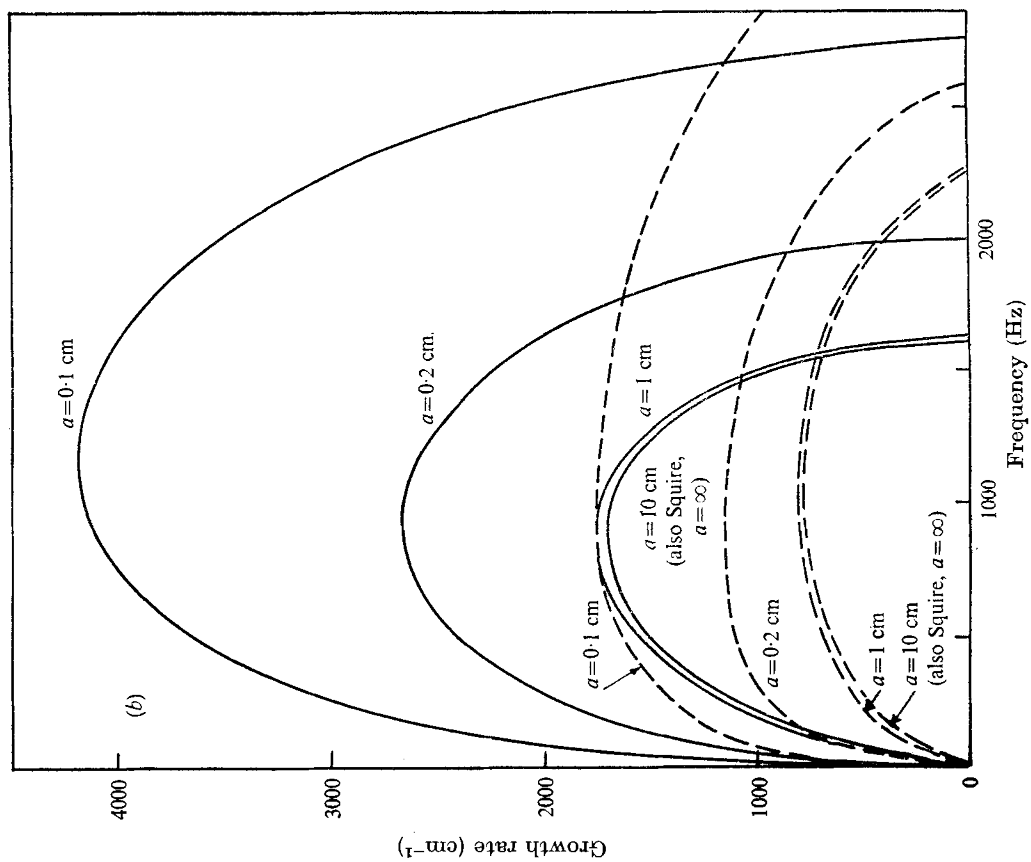


FIGURE 2. Characteristics of (*a*) symmetric and (*b*) antisymmetric waves (V = 1000 cm/s). ——, $h = 10^{-3}$ cm; – – –, $h = 10^{-2}$ cm.

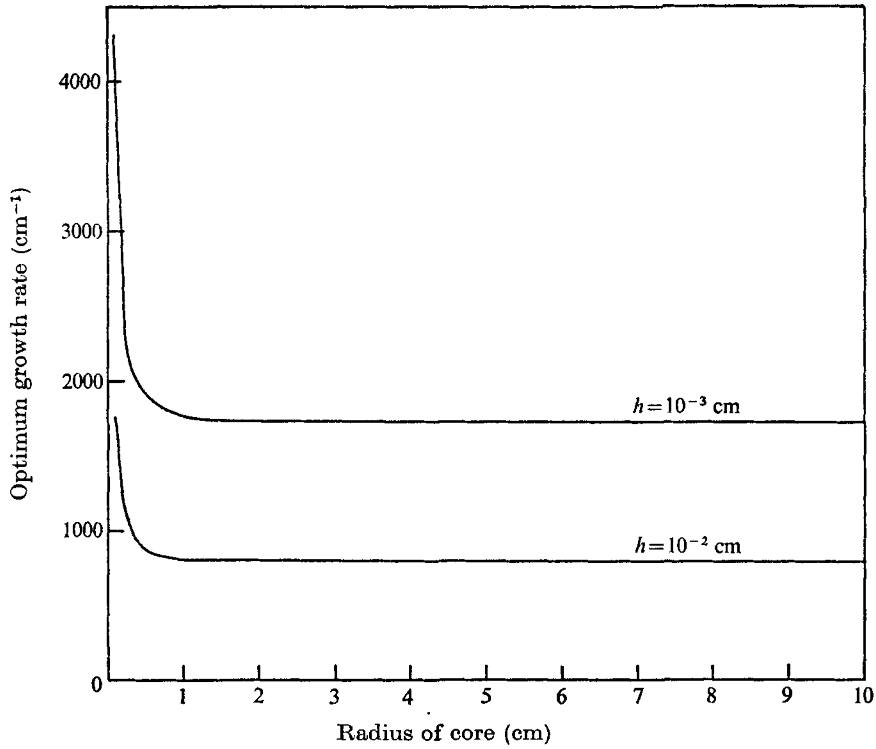


FIGURE 3. Variation of optimum growth rate with radius of core for antisymmetric waves (V = 1000 cm/s).

where

$$\left.\begin{aligned} P &= \Delta[I_1(ka)\,K_0(k(a+h)) + K_1(ka)\,I_0(k(a+h))],\\ Q &= \Delta[I_1(k(a+h))\,K_0(ka) + K_1(k(a+h))\,I_0(ka)],\\ R &= \Delta/k(a+h),\\ S &= \Delta/ka,\\ \Delta &= k^{-1}\{K_1(ka)\,I_1(k(a+h)) - I_1(ka)\,K_1(k(a+h))\}^{-1}.\end{aligned}\right\} \tag{8}$$

For large ka and $k(a+h)$ the quartic equation (7) factorizes into two quadratics, which are Squire's equations for antisymmetric and symmetric waves on a flat sheet:

$$\rho_L(Vk-\omega)^2\frac{1}{k}\tanh\left(\frac{kh}{2}\right)+\frac{\rho_G\omega^2}{k}-\sigma k^2, \tag{9}$$

$$\rho_L(Vk-\omega)^2\frac{1}{k}\coth\left(\frac{kh}{2}\right)+\frac{\rho_G\omega^2}{k}-\sigma k^2. \tag{10}$$

Equation (7) has been solved numerically for a range of values of the sheet velocity V, sheet thickness h and cylinder radius a. Typical results for the growth rates of both symmetric and antisymmetric waves, and the corresponding wave velocities, are given in figures 2–4 for a sheet velocity of 1000 cm/s and a range of sheet thicknesses and radii. Curves plotted on the basis of Squire's analysis for

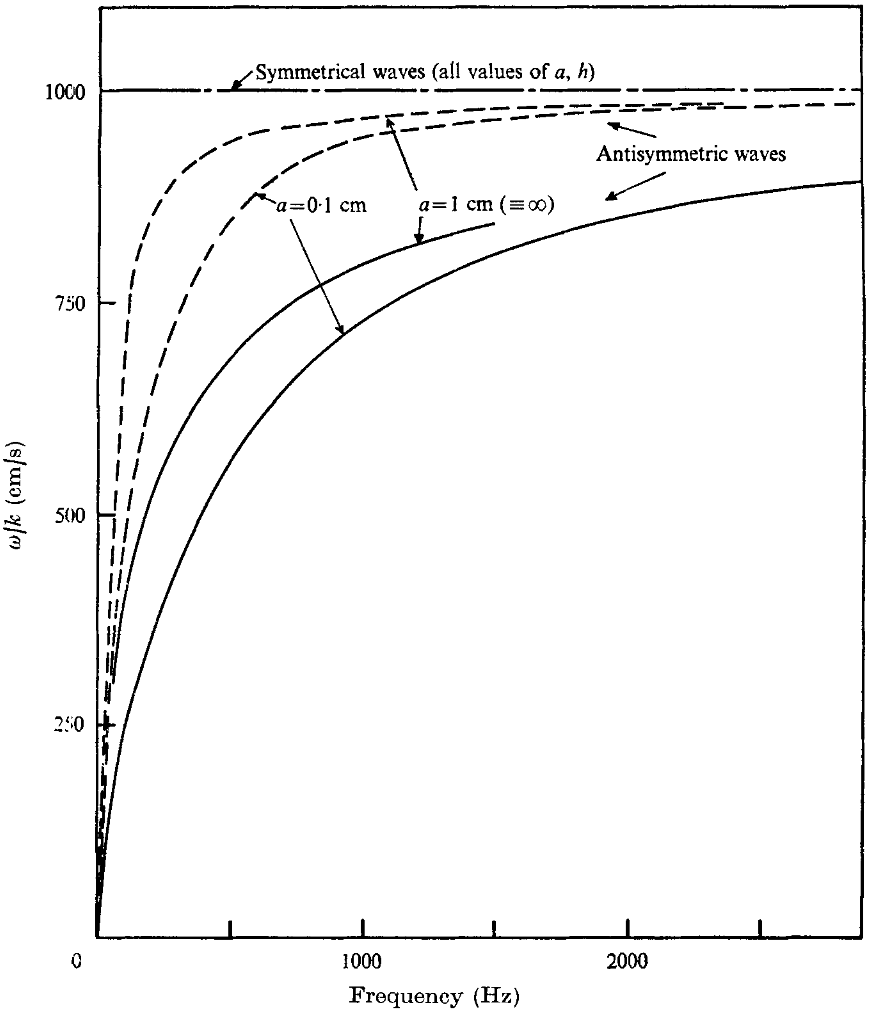


FIGURE 4. Wave velocities (V = 1000 cm/s). ——, $h = 10^{-3}$ cm; – – –, $h = 10^{-2}$ cm.

flat sheets [(9) and (10)] are included for comparison, although it should be noted that they coincide with those drawn for the larger values of a.

It is seen (figures 2a, b) that, for core radii greater than about 1 cm, the wave characteristics correspond to Squire's solutions. However, at lower values, curvature has a marked influence on the growth rate and range of instability, both increasing with reduction of the radius. The effect on the growth rate is brought out more clearly in figure 3, which plots the optimum value *vs.* core radius for antisymmetric waves. Some significant differences between the two types of waves can however be observed. For example, figure 2(a) (symmetric waves) shows the optimum frequency to increase with sheet thickness, while figure 2(b) (antisymmetric waves) demonstrates the optimum value to be affected by both the radius and sheet thickness.

Figure 4 shows that the velocities of symmetric waves are unaffected by sheet curvature and thickness and that they move at the sheet velocity for all frequencies. Antisymmetric wave velocities, however, which are always less than that of the sheet, increase with increasing radius and diminishing thickness, although the differences become less marked with increasing frequency.

The results of the analysis are interesting in that they could provide a further explanation for the observed shortness of conical sheets, produced from swirl spray nozzles, compared with flat sheets. It was suggested by Clark & Dombrowski that this could be due to relatively large disturbances produced in the air core. This work indicates that increased growth rates, particularly near the apex of the cone, are a contributory factor.

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